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BRIEF COMMUNICATION

SOME SUPPLEMENTAL ANALYSIS CONCERNING THE VIRTUAL MASS AND LIFT FORCE ON A SPHERE IN A ROTATING AND STRAINING FLOW

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1. INTRODUCTION

The authors of the present communication have previously published a paper (Drew & Lahey 1987) concerning the question of objectivity associated with the virtual mass and lift forces on a sphere imbedded in a rotating and straining inviscid flow. In that paper, the virtual mass and lift forces were calculated, and the sum of these two forces was shown to be objective, as required by the Principle of Material Frame Indifference (Truesdell & Toupin 1963); however, the individual terms need not be objective.

Unfortunately, subsequent to the publication of this paper (Drew & Lahey 1987), it was pointed out to the authors (Acrivos 1987) that a fundamental error had been made in the derivation, thereby affecting the conclusion. In particular, the assumption that the vorticity ζ_i was identically zero in the rotating frame, leading to [15] in that paper, was incorrect. The correct result is that if the vorticity is *initially* zero, and the shear in the far field is small, then the vorticity remains small for some time, and the results are approximately valid. In addition, there is an inconsistency in notation in the previous paper defining the velocities in the various coordinate frames.

It is the purpose of this communication to amend the analysis to correct the error. Moreover, it will be shown in what sense the final results and conclusions reached in the previous paper are valid.

The velocity of the fluid far from the sphere is

$$v_{ci}^{*\infty} = v_{0i}^{*} + e_{ij}^{*} x_j^{*} + \epsilon_{ijk} \omega_j^{*} x_k^{*} \qquad [1]$$

[$v_{ci}^{*\infty}$ is simply called v_{ci}^{*} in Drew & Lahey (1987)]. The coordinates are rotated according to

$$x_i = Q_{ij} x_j^{*}, \qquad [2]$$

where the rotation tensor Q_{ij} is chosen to eliminate rotation from v_{ci}^{∞}, i.e.

$$\dot{Q}_{ij} = -Q_{ik} \epsilon_{kmj} \omega_m^{*}. \qquad [3]$$

This gives

$$v_{ci}^{\infty} = v_{0i} + e_{ij} x_j, \qquad [4]$$

where

$$v_{ci}^{\infty} = Q_{ij} v_{0j}^{*}$$

and

$$e_{ij} = Q_{ik} Q_{jl} e_{kl}^{*}.$$

2. EFFECTS OF VORTICITY

The equations of motion in a rotating frame (Greenspan 1968) are

$$v_{i,i} = 0 \qquad [5]$$

$$v_{i,t} - \epsilon_{ijk} v_j \zeta_k + 2\epsilon_{ijk} \omega_j v_k = -P_{,i}, \qquad [6]$$

where $\zeta_i = \epsilon_{ijk} v_{k,j}$ is the vorticity and

$$P = \frac{p}{\rho_c} + \tfrac{1}{2} v_i v_i - \tfrac{1}{2}\omega^2 (x_1^2 + x_2^2). \qquad [7]$$

The Helmholtz representation for the velocity is

$$v_i = \phi_{,i} + \epsilon_{ijk} A_{k,j} = \phi_{,i} + v_i' \qquad [8]$$

where ϕ is the velocity potential, $v_i' = \epsilon_{ijk} A_{k,j}$, and A_k is a vector field with $A_{k,k} = 0$. Hence the vorticity ζ_i is given by

$$\zeta_i = -A_{i,jj}. \qquad [9]$$

The equation for the vorticity can be derived by taking $\epsilon_{ijk}(\ )_{j,k}$ of the momentum equation [7]. We have

$$\zeta_{i,t} + v_j \zeta_{i,j} - (2\omega_j + \zeta_j) v_{i,j} = 0. \qquad [10]$$

Here we note that [unlike the assumption made in [15] in Drew & Lahey (1987)], $\zeta_i \not\equiv 0$. We further note that the vorticity ζ_i is determined from an evolution equation, so that an initial condition is needed for it. We take, as did Proudman (1916),

$$\zeta_i(\boldsymbol{x}, 0) = 0.$$

Thus, it appears that the previous analysis is correct "at the initial instant," but will be incorrect as the flow (and therefore the vorticity) evolves. We shall discuss this point further.

The continuity equation [6] gives

$$v_{i,i} = \phi_{,ii} + \epsilon_{ijk} A_{k,ji} = \phi_{,ii} = 0. \qquad [11]$$

The boundary condition on the surface of the sphere is

$$v_i n_i = v_{di} n_i, \qquad [12]$$

where n_i is the unit exterior normal to the sphere.

Thus, from [8] and [10],

$$v_i n_i = \phi_{,i} n_i + \epsilon_{ijk} A_{k,j} n_i = v_{di} n_i. \qquad [13]$$

Lighthill (1956) (see Auton 1987) gives the velocity v_i' as

$$v_i' = \iiint G_{ij}(x_k, x_k') \zeta_j(x_k', t) \, \mathrm{d}V', \qquad [14]$$

where Auton gives the general form for the kernel G_{ij}. The kernel represents the velocity field due to a point vortex, its image in the sphere and a line vortex in the sphere. From Lighthill's form, it is easily seen that

$$G_{ij}(x_k, x_k') n_i = 0 \quad \text{on} \quad |\boldsymbol{x} - \boldsymbol{x}_d| = R. \qquad [15]$$

Thus,

$$v_i' n_i = 0 \qquad [16]$$

so that

$$\phi_{,i} n_i = v_{di} n_i \qquad [17]$$

at $|\boldsymbol{x} - \boldsymbol{x}_d| = R$.

Far from the sphere the velocity should approach the undisturbed fluid velocity, so that

$$\lim_{|x - x_d| \to \infty} \phi_{,i} = v_{0i} + e_{ij} x_j \qquad [18]$$

and

$$\lim_{|x - x_d| \to \infty} A_k = 0. \qquad [19]$$

The velocity potential is then given by

$$\begin{aligned}\phi = & -v_{di}(x_i - x_{di}) \frac{1}{2} \left\{ \frac{R^3}{[(x_j - x_{dj})(x_j - x_{dj})]^{3/2}} \right\} \\ & + (v_{0i} + x_{dj} e_{ij})(x_j - x_{dj}) \left\{ 1 + \frac{R^3}{[(x_j - x_{dj})(x_j - x_{dj})]^{3/2}} \right\} \\ & + \tfrac{1}{2}(x_i - x_{di}) e_{ij} (x_j - x_{dj}) \left\{ 1 + \frac{R^3}{[(x_j - x_{dj})(x_j - x_{dj})]^{3/2}} \right\}. \end{aligned} \qquad [20]$$

Equation [20] is exactly the same as the velocity potential derived previously ([40]; Drew & Lahey 1987).

Now it is clear how to obtain the velocity field v_i from the initial and boundary conditions. First, the potential part is given for all times in terms of the boundary conditions by [20]. If the vorticity is known at all spatial points, at a given time, the rotational part of the velocity, v'_i can be obtained from [14]. The vorticity evolves according to [10]. This differential equation gives $\partial \zeta_i / \partial t$ in terms of ζ_i and v_i. This is a differential equation for the vorticity, which has the added complication that it involves a spatial integral of the unknown function. Even so, it could be used to obtain a numerical approximation to the solution at discrete time intervals. We further note that the equation is conservative. Thus, there is no need to prescribe boundary conditions on the sphere surface, since that it a material surface. Boundary conditions are needed at infinity where no perturbation vorticity enters the flow domain. Thus, in theory, the velocity is determined from the boundary conditions at infinity and at $|x - x_d| = R$.

To find the force on the sphere, we note that the force is given by

$$F_i = \iint_S p n_i \, dS. \qquad [21]$$

We proceed as in the Drew & Lahey (1987), although the definitions are slightly different. We define

$$\hat{P} = \frac{p}{\rho_c} + \phi_{,t} + \tfrac{1}{2} \phi_{,k} \phi_{,k} - \tfrac{1}{2} \omega^2 (x_1^2 + x_2^2). \qquad [22]$$

Note that

$$-\hat{P}_{,i} = v'_{i,t} + (\phi_{,j} v'_i)_{,j} + (\phi_{,i} v'_j)_{,j} + (v'_i v'_j)_{,j} + 2\epsilon_{ijk} \omega_j \phi_{,k} + 2(\omega_k A_k \delta_{ij} - \omega_i A_j)_{,j}. \qquad [23]$$

If we take $()_{,j}$ of [23], we have

$$-\hat{P}_{,ii} = 2(\phi_{,j} v'_i)_{,ji} + (v'_i v'_j)_{,ji}. \qquad [24]$$

The boundary condition on $|x - x_d| = R$ is

$$-n_i \hat{P}_{,i} = n_i (\phi_{,i} v'_j)_{,j} + n_i (v'_i v'_j)_{,j} + 2\epsilon_{ijk} n_i \omega_j \phi_{,k} + 2 n_i (\omega_k A_k \delta_{ij} - \omega_i A_j)_{,j}. \qquad [25]$$

If v'_i and A_i are negligible, then $\hat{P}$ is the same as Drew & Lahey's (1987) [12], [19], [20] and [21]. From [14], we see that if $\zeta = 0$ then $v' = 0$. Thus, at the initial instant, the previous result holds. The previous result is approximately true when ζ is sufficiently small. For example, Auton (1987) calculates the perturbations to the velocity field due to a small rotation far from the sphere. The smallness that he assumes is sufficient for our analysis in the case that there are no fluid accelerations. However, we see from [10] that, in general, a perturbation vorticity that is initially zero will grow to be comparable to the mean fluid vorticity ω in a time of order R/U, where U is a velocity scale for the relative motion. We further note that Auton's result breaks down under the same conditions, i.e. when the vorticity is no longer small. Thus, the result of Drew & Lahey (1987) is equivalent to Auton's, in the special case of small vorticity. Furthermore, the present analysis shows the conditions under which both analyses break down.

3. CONCLUSION

Proceeding as in Drew & Lahey (1987) we obtain [45] of that paper, except now it is recognized that the expression is valid only for small vorticity. In this case, the combined virtual mass and lift forces are given by

$$\tfrac{4}{3}\pi R^3\rho\left[\frac{\mathrm{D_c}v^*_{\mathrm{c}i}}{\mathrm{D}t}-\frac{\mathrm{D_d}v^*_{\mathrm{d}i}}{\mathrm{D}t}-(v^*_{\mathrm{c}i,l}-v^*_{\mathrm{d}i,l})(v^*_{\mathrm{c}l}-v^*_{\mathrm{d}l})\right]. \quad [26]$$

As was noted previously, this combination of terms is objective, while the individual parts of it are not.

It is hoped that the error made in the previous paper (Drew & Lahey 1987) has not led to any confusion. In any event, this paper shows in what sense the final results and conclusions are valid, namely they are only approximately valid, as long as the fluid vorticity is small.

We note that the analysis presented here, involving an exact solution for the irrotational part of the flow and rigid body rotation as the approximate solution to the rotational part, is simpler than that given by Auton (1987), where a perturbation analysis is needed to compute the force. This is analogous to the situation encountered in calculating inertial effects in creeping flow, where the solution for inertialess flow around a body can be used in an integral to obtain an approximation for the inertial force (Brenner 1961; Brenner & Cox 1964).

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